Uwe Meixner · Albert Newen (Hrsg.)

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Buchbesprechungen

Book Reviews

Vincent Hendricks, Stig A. Pedersen, Klaus F. Jørgensen (Eds.): *Proof Theory – History and Philosophical Significance*

(Synthese Library 292)
Dordrecht: Kluwer Academic Publishers 2000

The book under review contains a collection of papers presented at a conference under the same title held at the University of Roskilde in 1997. It comprises 9 papers in 4 sections and an introduction by the editors:

0. Hendricks, Pedersen and Jørgensen: Introduction, 7 pages

Part 1. Review of Proof Theory
1. Solomon Feferman: Highlights in Proof Theory, 20 pages

Part 2. The Background of Hilbert's Proof Theory
2. Leo Corry: The Empiricist Roots of Hilbert's Axiomatic Approach, 19 pages
3. David Rowe: The Calm Before the Storm: Hilbert's Early Views on Foundations, 38 pages
4. Wilfried Sieg: Toward Finitist Proof Theory, 19 pages

Part 3. Brouwer and Weyl on Proof Theory and Philosophy of Mathematics
5. Dirk van Dalen: The Development of Brouwer's Intuitionism, 35 pages
6. Moritz Epple: Did Brouwer's Intuitionistic Analysis Satisfy its own Epistemological Standards? 25 pages
7. Solomon Feferman: The Significance of Weyl's Das Kontinuum, 15 pages
8. Erhard Scholz: Herman[sic!] Weyl on the Concept of Continuum, 22 pages

Part 4. Modern Views and Results From Proof Theory

9. Solomon Feferman: Relationships between Constructive, Predicative and Classical Systems of Analysis, 15 pages

We will give short reviews of all of these papers including the introduction and then finish with some general remarks on the book.

0. Introduction by Vincent Hendricks, Stig Andur Pedersen and Klaus Frovin Jørgensen.

The introduction gives a first account of the topic of the book. It stresses the role of David Hilbert who founded Proof Theory as a subdiscipline of mathematical logic in the beginning of the 20th century. It refers to the well-known 'Grundlagenstreit' with Brouwer and the role of Weyl in the discussion.

It is the clear intention of the editors to defend proof theory as an important part of present philosophy of mathematics. This should be done by showing "the emergence of proof theory in a broad historical and philosophical context" and by tracing "the development of proof theory and illuminate its growth into a mature theory". Each paper of the volume is shortly summarized and put in its historical context. From this point of view, the introduction can be read as a very comprehensive abstract of the whole book.

In the reviewer's view, this introduction stresses indeed the crucial points of the papers with respect to the essence of proof theory. Therefore, we consider it a successful opening of the book. However, whether the editors indeed reach theirs goals with this volume is another question which we will address at the end of this review.

Part 1: Review of Proof Theory

1. *Hightlights [sic!] in Proof Theory* by Solomon Feferman.

This is to my knowledge the best comprehensive presentation of the basic results in proof theory. It starts with a short review of Hilbert's programme and the idea of finitary proof theory. The second paragraph presents Gentzens sequent calculus with its applications. In the third paragraph, infinitary proof theory is introduced together with its application in the analysis of Peano Arithmetic. It finishes with the definition of the proof-theoretic ordinal of a formal system. The final paragraph contains further considerations about infinitary proof theory.

In his paper, Feferman has restricted himself to the methods and results developed up to the 1960s. However, it covers the basics of proof theory which are a must for every philosopher who is interested in the foundations of mathematics. To state it clear: nobody who does not understand the material presented in this paper, can take part in the debate of the foundations of mathematics. The paper requires some elementary knowledge of mathematical and logical notation, but it should be readable for philosophers, too.

Part 2: The Background of Hilbert's Proof Theory

2. *The Empiricist Roots of Hilbert's Axiomatic Approach* by Leo Corry

The paper of Corry gives a very interesting account to the background of Hilbert's seminal book "Grundlagen der Geometrie". For everybody interested in this topic, the paper probably contains useful information. It puts the Grundlagen in the context of the work of *Heinrich Hertz* (1857–1894), in particular his book "Die Prinzipien der Mechanik in neuem Zusammenhange", Leipzig 1894, *Carl Neumann* (1832–1925) and *Paul Volkmann* (1856–1938) [misspelled "Volkman" in the subtitle of page 42]. As the title suggests, Corry argues that Hilbert's view of geometry has some direct empiricist roots which can be found, for instance, in the early lectures of Hilbert about geometry. Then Corry is following Hilbert's arguments about the nature of the (geometric) axioms up to the well-known correspondence with Frege and the sixth problem of the axiomatization of physics in Hilbert's famous problem list of 1900.

In the concluding remarks Corry admits that "[c]learly there is a significant distance separating his [Hilbert's] early axiomatic work and, say, his work on proof theory". In fact, we can fully agree with this reservation. As we have already stated, we consider the paper as very informative about the history of the Grundlagen der Geometrie. However, if we ask how it relates to proof theory, the answer could be at best that the Grundlagen der Geometrie are not directly related to Hilbert's invention of proof theory.

3. *The Calm Before the Storm: Hilbert's Early Views on Foundations* by David Rowe

The paper of Rowe focuses even more than Corry on Hilbert's personal story as it helps to understand the genesis of the Grundlagen der Geometrie. In the first paragraph about Hilbert's early career, one can learn a lot about Hilbert's studies, his teachers and the academic environment. The second paragraph draws a detailed historical picture of the background of the Grundlagen der Geometrie. The third paragraph is devoted to Hilbert's axiomatic method and Frege's critique. Since Frege's critique was directly addressing the axioms of the Grundlagen, the axiomatic method is discussed mainly in relation with geometry. As Corry, Rowe has the problem to relate Hilbert's work on (the axiomatization of) geometry with proof theory. In the appropriate historical order, he is discussing the problems raised by the set-theoretical paradoxes following the Hilbert-Frege controversy about the status of the axioms in the Grundlagen. However, Hilbert is discussing (in public) the paradoxes in 1904 only, when he gave his talk at the International Congress of Mathematicians in Heidelberg. Rowe is following Hilbert's work in the foundations of *arithmetic* which became then a major topic. This topic is continued in the forth (and last) paragraph about "Hilbert's Return to Foundations" which Rowe marks at 1917.

This paper is very informative with respect to the personal relationships of Hilbert which presumably influenced him in his scientific career. However, as for Corry we are skeptical about the relation of Hilbert's work on the Grundlagen with proof theory. Although Rowe discusses the foundational work from 1905 on, the paper does not provide material to fill the gap between his work on the foundations of geometry and arithmetic.

4. *Toward Finitist Proof Theory* by Wilfried Sieg

As mentioned in the first endnote this paper is "a summary of developments analysed in [Sieg 1997a]", a then unpublished technical report of Carnegie Mellon University. In the meanwhile, the full version is published under the title "Hilbert's Programmes: 1917–1922" in the Bulletin of Symbolic Logic (Vol. 5, No. 1. pp. 1–44). However, the version published in this volume is still very valuable in particular if one does not like to read all 44 pages. It gives a very good account of the devolopment of the proof theoretic programme, or as it often called “Hilbert’s programme” by Hilbert and Bernays between 1917 and 1922. Apart from the published papers of the time, Sieg is using the most reasonable – and available – source of information: the unpublished lecture notes of lectures concerning foundational issues which Hilbert gave at that time. They are easily accessible at the library of the Mathematical Institute in Göttingen and they contain a lot of information about the emergence of ideas and methods which are finally presented in the seminal book on foundations of mathematics, the “Grundlagen der Mathematik” by Hilbert and Bernays (2 volumes, Berlin 1934 and 38, ^{2}1968 and 1970). The article finishes with a paragraph called “Remarks and Issues” which we consider as particularly valuable in this collection since for the first time the second part of the subtitle, the “Philosophical Significance” of Proof Theory is addressed. After a more general discussion, Sieg focusses on the following problem, phrased in his words: “If we take the expansion of the domain of objects for finitist considerations seriously, we are dealing not just with numerals, but more generally with elements of inductively generated classes.” And if one looks at the proof-theoretic investigations of inductive generated classes the philosophical significance of proof theory for mathematics as a whole should become obvious.

Part 3: Brouwer and Weyl on Proof Theory and Philosophy of Mathematics

5. *The Development of Brouwer’s Intuitionism* by Dirk van Dalen

Of course, in a collection about the philosophy of proof theory, Brouwer’s name cannot be missing. And with van Dalen the leading expert on the work and life of Brouwer is contributing. It is no surprise that the paper is one of the best articles of the volume giving a profound account of Brouwer’s development of intuitionism. Van Dalen is following a strict historical order, starting with Brouwer’s dissertation which contained already the main outline of the intuitionistic programme. In particular, he is referring to the “rejected” parts of the disseration, i.e, parts which Brouwer left out in the final version on the demand of his supervisor Korteweg. In fact, these parts are the more philosophical ones which are, nowadays, particularly valuable to understand Brouwer’s philosophy. The first two sections, called “Mysticism and the Dissertation” and “Issues and Topics in the Dissertation” are directly related to the disseration, and subsection 2.4. “Criticism of the axiomatic method and consistency methods” outlines Brouwer’s criticism of Hilbert’s proof theory (on the base of the “Heidelberg” paper: D. Hilbert. Über die Grundlagen der Logik und der Arithmetik. Verhandlungen des Dritten Internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 13 August 1904, S. 174–185, 1905). Brouwer’s view of the role of Logic in Mathematics is discussed in the section “Place and Function of Logic”. In section 4 he turns to the introduction of choice sequences. For it the reader needs some mathematical background – or, the other way around – he can learn a little bit of intuitionistic mathematics in formal terms, in particular with respect to the continuity theorem and the fan theorem. The final section, called "The Impact of the Grundlagenstreit" briefly addressed Brouwers response to Hilbert in his paper "Intuitionistische Betrachtungen über den Formalismus" (Die Preussische Akademie der Wissenschaften. Sitzungsberichte. Physikalisch-Mathematische Klasse, S. 48–52, 1928.) After the Annalenstreit (for which we like to refer, as the author, to his article "D. van Dalen. The War of the Frogs and the Mice, or the Crisis of the Mathemaische Annalen, Mathematical Intelligencer, 12, p. 17–31, 1990") Brouwer "withdrew from the foundational debate". However, van Dalen is referring to "a few isolated papers [...] which show that he had pursued intuitionistic mathematics privately."

Van Dalen's article gives the reader a very valuable overview of Brouwer's work. It draws the reader's attention to the rejected parts of the dissertation which are important for the philosophical background.

6. *Did Brouwer's Intuitionistic Analysis Satisfy Its Own Epistemological Standard?* by Moritz Epple

The article of Epple starts slightly dissatisfactory: It opens with a remark that "on the *historical* level" the fan theorem will be discussed, while "on the *philosophical* level, I cannot do much more than raise the question formulated in the title of the essay." At the first glance one would expect, in the line of the papers of Corry and Rowe, a (hopefully interesting) historical paper which, however, does not contribute to the question of the philosophical relevance. But, in fact, the opposite is the case. A citation from the last paragraph of the third section gives what we consider as the key lesson from this essay: "The overall result of our discussion is therefore clear. What Brouwer presented as a proof of his 'Haupttheorem der finiten Mengen' was, in effect, the implicit introduction of a new proof scheme or postulate into the foundations of intuitionistic analysis." This lesson which we consider as highly relevant for the philosophical significance of proof theory, is given by a formal treatment of the fan theorem. This requires some technical work which is clearly due to the nature of the problem. However, if one masters this, the prize is a deeper insight in the (one) philosophical problem(s) of intuitionistic analysis. The "Introduction" of the paper confronts the reader already with the necessary technicalities to state the fan theorem. The second section gives "Some comments on the epsitemological standards of intuitionism". Section 3 is called "The problem of solving the fan theorem" and is, again, more technical. The following section provides an "Epistemological discussion". The paper finishes with a "Historical discussion" and a "Conclusion".

Although we cannot follow all arguments of the author, in particular his clear emphasis on the epistemological problem of (the intuitionistic proof of) the fan theorem, we consider this contribution as one which, indeed, shows the philosophical significance of proof theory. And, "the question formulated in the title" deserves further philosophical discussion.

7. *The Significance of Weyl's Das Kontinuum* by Solomon Feferman

In his second contribution to this volume, Feferman presents a short introduction to Hermann Weyl's work on foundational issues. In particular, he is referring to three important contributions of Weyl: the early paper "Über die Definitionen der mathematischen Grundbegriffe" from 1910 (Math.-nat. Blätter, 7: 93–95, 109–113), the monograph "Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis" (Teubner, 1918) and the well-known paper "Über die neue Grundlagenkrise der Mathematik" (Math. Zeitschrift, 20: 131–150, 1921). In the first part of his article, Feferman points out that the first paper is "sadly overlooked, because what he [Weyl] does there is provide an explanation of – what we call in modern terms – the notion of definability over any relational structure, and which anticipates Tarski's famous contribution in that respect." Then, a short overview of the shifts of Weyl's view on the foundations is given by the following table:

1. 1910 Contribution to Zermelo's set theory
2. 1917 Critical of set theoretical foundations
3. 1918 Definitionism (à la Poincaré)
4. 1920 Joins Brouwer's intuitionistic programme and criticizes Hilbert's programme
5. 193? Gradual disillusionment with intuitionism
6. 193? Reaffirmation of value of 1918 contribution
7. 1953 Turn between "constructivity" and "axiomatics"

The core of the paper is the discussion of Weyl's mathematical framework presented in his monograph "Das Kontinuum". After a crash course about Russell's Ramified Theory of Types, Feferman presents a reconstruction of Weyl's systems in modern terms. From this perspective, it can be identified with the system $\mathbf{ACA}_0$ – Arithmetical Comprehension (with restricted induction on the natural numbers). This system is a well-known theory within the programme of "Reverse Mathematics" (cf. S. Simpson, Subsystems of Second-Order Arithmetic, Springer, 1999). Feferman discusses in particular the Least Upper Bound Axiom which can be saved in Weyl's system only in a version for sequences (rather than sets). The paper finishes with a list of the examples of theorems which can be formalized in $\mathbf{ACA}_0$ and the limitations of the system. A very elaborated reconstruction of Weyl's "Das Kontinuum" can be found in another paper of Feferman to which he refers several times: Weyl vindicated: *Das Kontinuum* 70 years later, in: Temi e prospettive della logica e della filosofia della scienza contemporanee, I, Bologna, p. 59–93, 1988; reprinted in: Feferman, In the Light of Logic, Oxford University Press.

This contribution is very valuable, on the one hand, because of its "rediscovery" of Weyl's paper of 1910. On the other hand, because it gives the reader a translation of Weyl's system within a modern formalism. And it shows that the proof theory of today provides the appropriate tools to investigate informal theories of the early 20th century.

In the middle of the paper, sometimes the index $_0$ for $\mathbf{ACA}_0$ got lost which is obvious in the line: "in its most general form in **ACA** (or even **ACA**)." Here the first one should probably come with the index, as well as all other following occurences of **ACA**.

8. *Herman[sic!] Weyl on the Concept of Continuum* by Erhard Scholz

Scholz discusses Weyl's concept of the continuum mainly with respect to his contributions to physics. Of course, this is a major motivation for the mathematical work of Weyl – often overlooked or even just not understood by logicians. Scholz shows how Weyl was struggling with an intuitionistic (or semi-intiuitionistic) reconstruction of the continuum, understood as the *real, physical continuum.* However, despite of the obvious relation of these questions to Weyl's foundational interests, the author has problems to relate the rather technical work of Weyl in topology and differential geometry with basic *proof-theoretic* questions (at least in our opinion).

The only question which is implicitly posed is whether a *mathematical symbol system* can capture *external reality*. Therefore, the paper is slightly off-topic with respect to the aim of the book. But with respect to Weyl's work in the foundations of space we consider this contribution as very interesting and useful. And, as a logician, one should have in mind that for the mathematicians of the early 20th century the relation (applications) of mathematics to physics were the true motiviation of their work, definitely more important than pure foundational interests.

Part 4: Modern Views and Results from Proof Theory

9. *Relationships between Constructive, Predicative and Classical Systems of Analysis* by Solomon Feferman

In his third contribution, Feferman presents "some redevelopments of classical analysis on both constructive and predicative grounds, with an emphasis on modern approaches". Both are given first "from a more informal, mathematical, point of view [...] and then from a formal, metamathematical point of view."

The first section gives an impressively clear account of Errett Bishop's constructive approach to analysis. In contrast to Brouwer, Bishop did not change the meaning of the mathematical vocabulary (which, obviously, is one of the reasons for the reservations which mathematicians had/have against intuitionism), but his work "can be read as a part of classical analysis, though developed in more refined terms." Bishop's criticism of classical mathematics relies on its "deficiency in numerical meaning", i.e, "that if you say something exists you ought to be able to produce it". Feferman gives a short introduction to the basic notions of constructivism, first of all, of its account of the real numbers, where Cauchy sequences are associated with *modulus-of-convergence-functions*. For the predicative redevelopment of analysis, Feferman refers to the system **W** (for "Weyl") introduced by him in the paper "Weyl vindicated" (cited above). He states the conjecture that "All (or almost all) scientifically applicable analysis can be carried out in **W**." However, for a deeper support of this thesis one would have to consult other papers of Feferman. In the metamathematical part, he proposes formal systems for the formalization of both Bishop style constructive mathematics and predicative mathematics. For the former one, he is mainly referring to his own framework $\mathbf{EM}_0|$(cf. e.g. Feferman, Constructive theories of functions and classes, in: Logic Colloquium '78, p. 87–139, North-Holland, 1979). For the latter one, he uses $\mathbf{RA}_{<\Gamma 0}$ as reference system and discusses the *predicative reducibility* of the systems $\mathbf{ACA}_0$, **W** and $\mathbf{WKL}_0$ ("Weak König's Lemma", another theory within "Reverse Mathematics").

We consider the presentation of Bishop style constructive mathematics in the first part of the paper as one of the most important contributions with respect to the aim of the volume. At least with respect to the *mathematical significance* of proof theory, it is nowadays much more influential than the intutionistic programme of Brouwer. And we are strongly convinced that the mathematical significance immediately implies *philosophical significance*. For the rest of the paper, however, it would be very helpful for the reader to know something in advance about the formal systems discussed in the programme of *Reverse Mathematics* (cf. Simpson's book cited above) or Feferman's framework of *Explicit Mathematics*. In general, Beeson's book "Foundations of Constructive Mathematics", Springer, 1985, serves as an excellent reference for further reading.

General remarks

All in all, the volume under review is a very good book. Taken isolated, all contributions are interesting papers which contain valuable information for different kinds of readers. We can recommend the book to everybody interested in the history of proof theory, despite the fact that the price of the book allows probably only for a look whether it is available in the local library. However, with respect to the title: "Proof Theory – History and Philosophical Significance", in particular with respect to the second part of the subtitle, we have some doubts whether the editors reached their original goal. Let us give a reordering of the papers with respect to their main orientation:

1. Mathematical proof theory
   a. Feferman, three times
   b. Van Dalen (partly)
   c. Epple (in some aspects)
2. History
   a. Corry
   b. Rowe
   c. Sieg
   d. Van Dalen
   e. Scholz (in parts)

3. Philosophical significance
   a. Sieg (in some aspects)
   b. Epple
   c. Feferman (from an abstract point of view)

As we have already pointed out, we consider Scholz' paper as somewhat being outside of the scope of this book. Nevertheless it gives interesting historical and philosophical information about Hermann Weyl, an important contributor to proof theory.

The most valuable papers are clearly the three lectures of Feferman. They are actually transcripted from audio tapes and transparencies of the lectures given at the conference in Roskilde. They would serve as an excellent starting point for a course on (constructive/predicative aspects of) proof theory. Nevertheless they are first of all mathematical papers, although they contain historical information with respect to the development of proof theory, and the philosophical significance should become obvious from the presented results on (meta)mathematics. Van Dalen's paper gives a comprehensive account to the whole life and work of Brouwer. Due to his important role in proof theory, this paper also fits perfectly the main aim of the volume. The same can be said for Sieg's contribution which is focussing on Hilbert and Bernays in the crucial years of 1917 to 1922. Finally, we consider Epple's article as one which is bringing up an interesting question related to the philosophical significance of proof theory.

In contrast, the papers of Corry, Rowe and Scholz – all of them very informative and interesting to read – are slightly outside the scope of the original aim of the book. Even if one considers historical investigations about the people involved in the development of a field as relevant for the history of this field (and we do so!), the first two miss the topic by concentrating too much on Hilbert's Grundlagen der Geometrie, while the last one seems to be too much oriented to the work of Weyl in physics and geometry.

It cannot be expected that every article is just in the absolute core of the title. Therefore, we consider these "extensions" not as a real problem. However, what has to be considered as a deficit is what is missing in the volume: if one likes to argue for the philosophical significance of proof theory, it is hardly acceptable to leave out Gödel's work. There is no doubt that many of the contributions of Gödel (by far not only his completeness and incompleteness theorems!) are proof-theoretic in nature – let's think of the double negation interpretation, the functional interpretation and his work on the constructible hierarchy. All these topics are still very vital in present research in logic. We stress this point since there is danger that one might take the given volume as evidence for the assumption that proof theory is mainly a "historical" field which was discussed in the first part of the last century and which had its philosphical significance just in the early disput about formalism and intuitionism. This is hardly the case, and the modern developments in (Bishop style) constructive mathematics (only addressed by Feferman in the last paper) are counterexamples to such a claim. This has to be maintained in particular against the so-called "Maverick philosophers" (whoever they are) addressed by the editors in the introduction. In fact, we would like to take over the final words of Sieg's article which indicate that we are only at a starting point of a philosophical discussion of proof theory: "… let me emphasize that there is a mine for historical, logico-mathematical, and philosophical investigations: join in!"

Reinhard Kahle, Universität Tübingen and Universidade Nova de Lisboa